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GEOMETRIC GAUGE THEORY OF METRIC DEFECTS

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Abstract—The mechanical behavior of a material manifold with dislocations and disclinations is explored by applying non-Riemannian geometry and gauge field theory. A geometric gauge theory of metric defects is introduced by local Lorentz invariance. As a result, we give the connection coefficients with the affine and the gauge connection. Taking the displacement field, the frame field and gauge field as basic parameters, we obtain the constitutive equations and the governing equations based on a variational principle with respect to the groups of a coordinate transformation and a gauge transformation.

1. INTRODUCTION

Generalized continuum mechanics is an important phase of current development in modern continuum mechanics. This field, initially studied by Kondo (1954a,b), Kondo and Ishizuka (1955), Kroner and Rieder (1956), Kroner (1958), Bilby *et al.* (1955) and Bilby (1960), is closely related to the theory of non-Riemannian geometry. In continuous distribution theory of defects, it has been discovered that the reference configuration, in the constructs of non-Riemannian space, such as metric, torsion and curvature tensors, is a Euclidean space with Euclidean metric structure and topological structure. According to the breaking of different structures of Euclidean space, defects are called metric or topological defects, respectively.

When the Yang–Mills (1954) theory was established, one recognized that Riemannian geometry itself essentially belongs to a kind of gauge field theory given by Utiyama (1956, 1971). Furthermore, it has recently been learned from the study of supergravity that the geometry of non-Riemannian space with non-vanishing torsion also belongs to a kind of non-Abelian gauge theory.

It is known that non-Abelian gauge theory can be naturally applied to any field in theoretical physics, provided that the field is related to Riemannian or non-Riemannian geometry. Based on this point of view, some work has been done in using the gauge theory to study generalized continua. Golebiewska-Lasota (1979) and Golbiewska-Lasota and Edelen (1979) first used Abelian gauge theory to discuss the guage invariance of the governing equations with electromagnetic field theory. Their work led to further study by Edelen (1980) and Kadic and Edelen (1983) in Yang–Mills minimal coupling for materials with dislocations and disclinations. In the field of geometric gauge theory, Duan (1985) and Duan and Duan (1986) discussed the geometric representation of the gauge theory of defects.

For a complete theory of generalized continuum mechanics, we have to deal not only with the geometric aspects of the material manifold but some process of physics. In this paper, we establish a geometric gauge theory of metric defects based on Lorentz invariance and continuous distribution theory of defects.

2. OBSERVATIONS OF THE MODELING

The mathematical theory of non-Abelian gauge theory, which was introduced by Yang and Mills (1954), takes the transformation of gauge potentials as

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$$B'_\mu = S^{-1} B_\mu S + \frac{i}{\varepsilon} S^{-1} \partial_\mu S \tag{1}$$

where S is the spin gauge transformation.

Kondo and Ishizuka (1955) gave the transformation of connection as follows:

$$\Gamma^\alpha_{\beta\lambda} = A^\alpha_i B^j_\beta B^k_\lambda \Gamma'^i_{jk} + A^\alpha_i \partial_\beta B^i_\lambda \tag{2}$$

which is symmetric or non-symmetric with respect to indices β and λ under the coordinate transformation

$$B^i_\beta = \frac{\partial x^i}{\partial X^\beta}, \quad B^{-1} = (A^\alpha_i). \tag{3}$$

From the expressions (1) and (2), we obtain the following information.

(1) These transformations have non-homogeneous terms $(i/\varepsilon)S^{-1}\partial_\mu S$ and $A^\alpha_i \partial_\beta B^i_\lambda$ under corresponding transformation. Therefore, (1) and (2) mean that the symmetry is broken.

It is well known that the transformation operator B^i_β may be written as

$$B^i_\beta = \delta^i_\beta + \nabla_\beta W^i \tag{4}$$

where

$$\nabla_\beta W^i = \partial_\beta W^i + \Gamma^i_{kl} W^k \partial_\beta W^l \tag{5}$$

are covariant derivatives with respect to the connections Γ^i_{kl}. Under the first order approximation, covariant derivatives $\nabla_\beta W^i$ may be rewritten as

$$\nabla_\beta W^i \overset{(\beta)}{=} \partial_\beta W^i. \tag{6}$$

Then we have

$$B^i_\beta = \delta^i_\beta + \partial_\beta W^i = \delta^i_\beta + \pi^i_\beta \tag{7}$$

where π^i_β are called differential extensions.

For fixed index j, π^j_β are covariant components of a vector in the (β) system and are written as the sum of a gradient and a rotation

$$\pi^i_\beta = a^i_\beta + c^i_\beta$$

based on the principle of decomposition, where

$$a^i_\beta = (\text{grad}\,\varphi^i)_\beta, \quad c^i_\beta = (\text{rot}\,d^i)_\beta.$$

Therefore

$$A^\alpha_i \partial_\beta B^i_\lambda = A^\alpha_i(\partial_\beta a^i_\lambda + \partial_\beta c^i_\lambda). \tag{8}$$

In a vector field, rot·grad ≡ 0, the Ricci coefficients

$$\Omega^\alpha_{\beta\lambda} = 2A^\alpha_i \partial_{[\beta} c^i_{\lambda]}. \tag{9}$$

It means that the antisymmetric field c^i_β plays a leading role and symmetric field a^i_β plays an indirect role. The non-symmetric part of the connection gives an antisymmetric field which is not only induced by a stress field, but can also be generalized to be induced by other physical effects. Therefore, symmetry breaking will play a role in an antisymmetric field.

(2) Since Lorentz group $\mathscr{L}$ is a linear transformation group depending on some parameters, the gauge symmetry of a rotation field will be broken under a local Lorentz transformation group and the role of a rotation field may be determined by the physical effect of a gauge field. Therefore, gauge potentials $B_\mu^{a'b'}$ must be antisymmetric for contravariant indices a' and b', i.e.

$$B_\mu^{a'b'} + B_\mu^{b'a'} = 0. \tag{10}$$

This means that gauge potentials $B_\mu^{a'b'}$ take values in Lie algebra $\mathfrak{l}$ of Lorentz group $\mathscr{L}$. The coordinate components of gauge potentials are $B_{\beta\lambda}^{\alpha}$, which satisfy the following transformation laws:

$$B_{\beta\lambda}^{\alpha} = A_i^{\alpha} B_\beta^j B_\lambda^k B_{jk}^i + A_i^{\alpha} \partial_\beta B_\lambda^i \tag{11}$$

and their antisymmetric parts are

$$B_{[\beta\lambda]}^{\alpha} = A_i^{\alpha} B_\beta^j B_\lambda^k B_{[jk]}^i + \Omega_{\beta\lambda}^{\alpha}$$

or

$$B_{[\beta\lambda]}^{\alpha} - \Omega_{\beta\lambda}^{\alpha} = A_i^{\alpha} B_\beta^j B_\lambda^k B_{[jk]}^i.$$

Let

$$S_{\beta\lambda}{}^{\alpha} = B_{[\beta\lambda]}^{\alpha} - \Omega_{\beta\lambda}^{\alpha} \tag{12}$$

then

$$S_{\beta\lambda}{}^{\alpha} = A_i^{\alpha} B_\beta^j B_\lambda^k S_{jk}{}^i \tag{13}$$

where

$$S_{jk}{}^i = B_{[jk]}^i, \quad \Omega_{jk}^i \equiv 0. \tag{14}$$

The expression (13) means that gauge potentials $B_{\beta\lambda}^{\alpha}$ are gauge-invariant and the role of an antisymmetric field is determined by both the antisymmetric part $B_{[\beta\lambda]}^{\alpha}$ of the gauge potentials and Ricci coefficients $\Omega_{\beta\lambda}^{\alpha}$.

(3) In the mathematical theory of a gauge field, we choose a torsion tensor $S_{\beta\lambda}{}^{\alpha}$ and a spin

$$S_{\beta\lambda}^{\alpha} = S_{\beta\lambda}{}^{\alpha} - S_{\beta}{}^{\alpha}{}_{\lambda} - S_{\lambda}{}^{\alpha}{}_{\beta} \tag{15}$$

as a gauge-invariant physical variable which is independent of the choice of coordinate system, where

$$S_{\beta}{}^{\alpha}{}_{\lambda} = g_\lambda^\mu B_{[\mu\beta]}^{\alpha} + g_\lambda^\mu \Omega_{\mu\beta}^{\alpha}$$

$$S_{\lambda}{}^{\alpha}{}_{\beta} = g_\beta^\mu B_{[\lambda\mu]}^{\alpha} + g_\beta^\mu \Omega_{\lambda\mu}^{\alpha}. \tag{16}$$

From (13) and (15), it is obvious that we must choose gauge potentials $B_{\beta\lambda}^{\alpha}$ as basic variables of a field.

3. THE PHYSICAL MODEL

In the classical theory of a continuum with defects, the dynamics of dislocations satisfied global Lorentz invariance, and defects in generalized solids satisfied only local Lorentz invariance in coupled physical fields. From the viewpoint of fields, just as the physical substance of a topological defect introduces an antisymmetric field, there are other physical fields having antisymmetry. Therefore, we treat them uniformly with the role of antisymmetric fields by gauge potentials having antisymmetry.

For the physical model of this paper, we introduce two basic assumptions as follows:

(1) the dynamics of continuum dislocations satisfies a local Lorentz invariance in generalized solids;

(2) first integrable conditions of a frame field are broken.

Thus, from assumption (1), there exists a local Lorentz frame field $\{e^{a'}_{\alpha}(x)\}$ which transforms on every space-time point, where $a' = 0, 1, 2, 3$ is the index of the frame and $\alpha = 0, 1, 2, 3$ is the index of the coordinate. For a', $e^{a'}_{\alpha}(x)$ are contravariant components of the local Lorentz frame and for α, they are covariant components of the local coordinate system, and they satisfy

$$e^{a'}_{\alpha} e^{\beta}_{a'} = \delta^{\beta}_{\alpha}, \quad e^{a'}_{\alpha} e^{\alpha}_{b'} = \delta^{a'}_{b'}.$$

$$\alpha, \beta = 0, 1, 2, 3; \; a', b' = 0, 1, 2, 3.$$

The Lorentz frame field determines the metric of space time

$$(\mathrm{d}S)^2 = g_{\alpha\beta}\, \mathrm{d}X^{\alpha}\, \mathrm{d}X^{\beta}; \; g_{\alpha\beta} = \eta_{a'b'} e^{a'}_{\alpha} \cdot e^{b'}_{\beta}$$

where $\eta_{a'b'} = \mathrm{diag}(1, -1, -1, -1)$ is the local Minkowski value of the metric and

$$g^{\alpha\beta} = e^{\alpha}_{0} \cdot e^{\beta}_{0} - e^{\alpha}_{1} \cdot e^{\beta}_{1} - e^{\alpha}_{2} \cdot e^{\beta}_{2} - e^{\alpha}_{3} \cdot e^{\beta}_{3}.$$

The matrix of the metric may be written as

$$G = (g_{\alpha\beta})_{0 \leqslant \alpha, \beta \leqslant 3} = (e^{a'}_{\alpha}) J (e^{b'}_{\beta})$$

where $J = (\eta_{a'b'})$. Therefore, the local transformation group of the frame is a Lorentz group $\mathscr{L}$.

From assumption (2),

$$\Omega^{a'}_{\beta\lambda} = \partial_{[\beta} e^{a'}_{\lambda]} = \gamma^{a'}_{b'c'} \cdot e^{b'}_{[\beta} \cdot e^{c'}_{\lambda]} \neq 0,$$

where $\gamma^{a'}_{b'c'}$ is a connection of the frame field $e^{a'}_{\alpha}(x)$, and

$$\gamma^{a'}_{b'c'} = e^{\alpha}_{b';\beta} \cdot e^{\beta}_{c'} \cdot e^{a'}_{\alpha} = \left[\partial_{\beta} e^{\alpha}_{b'} - \begin{Bmatrix} \alpha \\ \beta \quad \lambda \end{Bmatrix} e^{\lambda}_{b'} \right] e^{\beta}_{c'} \cdot e^{a'}_{\alpha} \neq 0$$

where $\partial_{\beta} = \partial / \partial x^{\beta}$.

We introduce a differential operator

$$X_{a'} = e^{\alpha}_{a'} \partial_{\alpha}.$$

For the covariant components of the frame, we have

$$2X_{[c'} e^{\beta}_{b']} = X_{c'} e^{\beta}_{b'} - X_{b'} e^{\beta}_{c'} = c^{a'}_{b'c'} e^{\beta}_{a'}$$

where

$$c^{a'}_{b'c'} = \gamma^{a'}_{b'c'} - \gamma^{a'}_{c'b'} = 2\gamma^{a'}_{[b'c']}$$

and

$$2X_{[c'}\gamma^{a'}_{\ b|d']} = X_{c'}\gamma^{a'}_{\ b\,d'} - X_{d'}\gamma^{a'}_{\ b\,c'}$$
$$= \gamma^{a'}_{\ f'[d'}\gamma^{f'}_{\ |b|c']} - \gamma^{a'}_{\ b\,f'}c^{f'}_{b'c'} - R_{b'c'\ d'}^{\ \ \ a'}$$

where $R_{b'c'\ d'}^{\ \ \ a'}$ are the frame components of a curvature tensor.

4. THE EUCLIDEAN CONNECTION

Let $\{e^{a'}_{\alpha}(x)\}$ be a Lorentz frame field in space-time of four dimensions and $V^{a'}_{\beta}$, $V^{a'}_{\beta}+\mathrm{d}V^{a'}_{\beta}$ be two vectors at point $P(x^{\beta})$ and its adjacent point $Q(x^{\beta}+\mathrm{d}x^{\beta})$, respectively. The Lorentz frame field introduces a change in $V^{a'}_{\beta}$ as

$$\delta V^{a'}_{\beta} = \mathrm{d}V^{a'}_{\beta} - \eth V^{a'}_{\beta} \tag{17}$$

where $\mathrm{d}V^{a'}_{\beta}$ is the change introduced by vector $V^{a'}_{\beta}$ itself, and $\eth V^{a'}_{\beta}$ is the change introduced by Lorentz frame field at points P and Q.

A matrix element of Lorentz group $\mathscr{L}$ is $L^{a'}_{a}$ at point P and $L^{a'}_{a}+\mathrm{d}L^{a'}_{a}$ at point Q, then

$$V^{a'}_{\beta}+\mathrm{d}V^{a'}_{\beta} = V^{a'}_{\beta}+L^{a'}_{a}\,\mathrm{d}V^{a}_{\beta}+V^{a}_{\beta}\,\mathrm{d}L^{a'}_{a}$$

and

$$V^{a'}_{\beta}+\eth V^{a'}_{\beta} = V^{a'}_{\beta}+L^{a'}_{a}\,\eth V^{a}_{\beta}+V^{a}_{\beta}\,\mathrm{d}L^{a'}_{a}.$$

Substituting (17) into the above two equations and subtracting them, we obtain

$$\delta V^{a'}_{\beta} = L^{a'}_{a}\delta V^{a}_{\beta}. \tag{18}$$

This means that $\delta V^{a'}_{\beta}$ is a vector and it is independent of the choice of the Lorentz frame.

For every vector $V^{a'}_{\beta}$, if

$$-\eth V^{a'}_{\beta} = \delta V^{a'}_{\beta} = 0 \tag{19}$$

holds, we say that the Lorentz frames at points P and Q are quasi-parallel.

Let $\Gamma^{\alpha}_{\beta\lambda}$ be the space time connection of the local coordinate system. For contravariant and covariant vectors $W^{\beta}_{a'}$ and $V^{a'}_{\beta}$, we have

$$\delta W^{\beta}_{a'} = \mathrm{d}W^{\beta}_{a'}+\Gamma^{\beta}_{\lambda\alpha}W^{\alpha}_{a'}\,\mathrm{d}x^{\lambda}$$
$$\delta V^{a'}_{\beta} = \mathrm{d}V^{a'}_{\beta}-\Gamma^{\alpha}_{\lambda\beta}V^{a'}_{\alpha}\,\mathrm{d}x^{\lambda}$$

and their covariant derivatives are

$$\nabla_{\lambda}W^{\beta}_{a'} = W^{\beta}_{a'|\lambda} = W^{\beta}_{a',\lambda}+\Gamma^{\beta}_{\lambda\alpha}W^{\alpha}_{a'}$$
$$\nabla_{\lambda}V^{a'}_{\beta} = V^{a'}_{\beta|\lambda} = V^{a'}_{\beta,\lambda}-\Gamma^{\alpha}_{\lambda\beta}V^{a'}_{\alpha} \tag{20}$$

where the comma indicates a partial derivative with respect to coordinates and the symbol "|" means a covariant derivative with respect to the connection $\Gamma^{\alpha}_{\lambda\beta}$.

We can obtain a transformation of the connection $\Gamma^{\alpha}_{\lambda\beta}$ with respect to non-holonomic transformation of the coordinates. On the one hand we have

$$\partial_{\lambda}e^{a'}_{\beta} = \Gamma^{\alpha}_{\lambda\beta}e^{a'}_{\alpha} = \Gamma^{\alpha}_{\lambda\beta}B^{i}_{\alpha}e^{a'}_{i}$$

and on the other hand

$$\partial_{\lambda}e^{a'}_{\beta} = \partial_{\lambda}[B^{j}_{\beta}e^{a'}_{j}]$$
$$= (\partial_{\lambda}B^{j}_{\beta})e^{a'}_{j}+B^{i}_{\beta}(\partial_{\lambda}e^{a'}_{j})$$
$$= (\partial_{\lambda}B^{j}_{\beta})e^{a'}_{j}+B^{j}_{\beta}B^{k}_{\lambda}(\partial_{k}e^{a'}_{j}).$$

Thus, we have

$$\Gamma^{\alpha}_{\lambda\beta} = A^{\alpha}_{i} B^{j}_{\beta} B^{k}_{\lambda} \Gamma^{i}_{kj} + A^{\alpha}_{i} \partial_{\lambda} B^{i}_{\beta}. \tag{21}$$

The frame field $\{e^{a}_{\alpha}(x)\}$ determines the metric and geometric structures of the space time. Therefore, indices of the frame get into all geometric objects and must be Lorentz invariant. Obviously, if we have

$$\Gamma^{a'}_{b'\beta} = \Gamma^{\alpha}_{\lambda\beta} e^{a'}_{\alpha} \cdot e^{\lambda}_{b'} = \Gamma^{\alpha}_{\lambda\beta} L^{a'}_{a} e^{a}_{\alpha} l^{b}_{b'} e^{\lambda}_{b} = \Gamma^{\alpha}_{\lambda\beta} e^{a}_{\alpha} e^{\lambda}_{b} L^{a'}_{a} l^{b}_{b'} = \Gamma^{a}_{b\beta} L^{a'}_{a} l^{b}_{b'},$$

we obtain

$$\Gamma^{\alpha}_{\lambda\beta} = \Gamma^{a'}_{b'\beta} e^{\alpha}_{a'} \cdot e^{b'}_{\lambda} \quad \text{and} \quad \Gamma^{\alpha}_{\lambda\beta} = \Gamma^{a}_{b\beta} e^{\alpha}_{a} \cdot e^{b}_{\lambda}. \tag{22}$$

Thus, on the one hand we have

$$\partial_{\lambda} e^{b'}_{\beta} = \Gamma^{\alpha}_{\lambda\beta} e^{b'}_{\alpha} = \Gamma^{\alpha}_{\lambda\beta} L^{b'}_{b} e^{b}_{\alpha} \tag{23}$$

and on the other hand we obtain

$$\begin{aligned} \partial_{\lambda} e^{b'}_{\beta} &= \partial_{\lambda} [L^{b'}_{b} e^{b}_{\beta}] \\ &= (\partial_{\lambda} L^{b'}_{b}) e^{b}_{\beta} + L^{b'}_{b} (\partial_{\lambda} e^{b}_{\beta}) \end{aligned} \tag{24}$$

in the local coordinate system. This means that $S_{\lambda\beta}{}^{\alpha}$ is a tensor and is gauge-invariant. $S_{\lambda\beta}{}^{\alpha}$ is called a torsion tensor.

From the isometric principle

$$-Q_{\mu\beta\lambda} = \nabla_{\mu} g_{\beta\lambda} = 0 \quad \text{or} \quad Q^{\beta\lambda}_{\mu} = \nabla_{\mu} g^{\beta\lambda} = 0, \tag{25}$$

we obtain the symmetric part of the Euclidean connection as

$$\Gamma^{\alpha}_{(\mu\lambda)} = \begin{Bmatrix} \alpha \\ \mu \quad \lambda \end{Bmatrix} - g^{\beta}_{\lambda} \Gamma^{\alpha}_{[\mu\beta]} - g^{\beta}_{\mu} \Gamma^{\alpha}_{[\lambda\beta]}; \tag{26}$$

then the Euclidean connection is

$$\Gamma^{\alpha}_{\mu\lambda} = \begin{Bmatrix} \alpha \\ \mu \quad \lambda \end{Bmatrix} + S_{\mu\lambda}{}^{\alpha} - S_{\mu}{}^{\alpha}{}_{\lambda} - S_{\lambda}{}^{\alpha}{}_{\mu} - \Omega^{\alpha}_{\mu\lambda} + \Omega^{\alpha}_{\mu\cdot\lambda} + \Omega^{\alpha}_{\lambda\cdot\mu}. \tag{27}$$

Let us now consider the gauge theory of metric defects under local Lorentz invariance. $B^{\alpha}_{\mu\lambda}$ is the coordinate component of the gauge potential and substituting (12) into (27) we obtain

$$\Gamma^{\alpha}_{\mu\lambda} = \begin{Bmatrix} \alpha \\ \mu \quad \lambda \end{Bmatrix} + B^{\alpha}_{\mu\lambda} - B^{\alpha}_{\mu\cdot\lambda} - B^{\alpha}_{\lambda\cdot\mu} \tag{28}$$

where

$$B^{\alpha}_{\mu\cdot\lambda} = g^{\beta}_{\lambda} B^{\alpha}_{\mu\beta} \quad \text{and} \quad B^{\alpha}_{\lambda\cdot\mu} = g^{\beta}_{\mu} B^{\alpha}_{\lambda\beta}. \tag{29}$$

Equation (28) shows that the frame field $\{e^{a}_{\alpha}(x)\}$ and the gauge potential $B^{\alpha}_{\beta\lambda}$ together determine the geometrical structure of the continuum with metric defects and the gauge field.

5. THE FIELD EQUATIONS

Firstly, we consider the structure of a Lagrangian function describing metric defects and a gauge field. Obviously, e^a_α and $B^{a\,b}_\mu$ are independent variables of the field describing the properties of a continuum with metric defects and a gauge field.

Setting

$$e^{\mu\nu} = (e^{a\mu\nu}_b)_{0 \leqslant a,b \leqslant 3} \tag{30}$$

$$e^{a\mu\nu}_b = \eta_{bc} e^a_{[\mu} e^c_{\nu]}, \tag{31}$$

we obtain the strength of a gauge field as

$$F^a_{b\mu\nu} = \partial_\nu B^a_{b\mu} - \partial_\mu B^a_{b\nu} + B^a_{c\nu} B^c_{b\mu} - B^a_{c\mu} B^c_{b\mu}. \tag{32}$$

and

$$F_{\mu\nu} = (F^a_{b\mu\nu})_{0 \leqslant a,b \leqslant 3}. \tag{33}$$

The strength of a gauge field satisfies the Bianchi equality

$$F^a_{b\mu\nu\|\lambda} + F^a_{b\nu\lambda\|\mu} + F^a_{b\lambda\mu\|\nu} = 0, \tag{34}$$

where the symbol "$\|$" means the covariant derivative with respect to the coordinate and the strength of a gauge field, i.e.

$$F^a_{b\mu\nu\|\lambda} = \partial_\lambda F^a_{b\mu\nu} - \begin{Bmatrix} \alpha \\ \lambda \quad \mu \end{Bmatrix} F^a_{b\alpha\nu} - \begin{Bmatrix} \alpha \\ \lambda \quad \nu \end{Bmatrix} F^a_{b\mu\alpha} + B^a_{c\lambda} F^c_{b\mu\nu} - B^c_{b\lambda} F^a_{c\mu\nu}. \tag{35}$$

Now the total Lagrangian function of the system is

$$L = L_e + L_g + L_{\text{int}}, \tag{36}$$

where L_e is the Lagrangian function of an elastic field, L_g the Lagrangian function of a gauge field and L_{int} the Lagrangian function describing the interaction between the frame field $e^{ab}_{\mu\nu}$ and the gauge field B^{ab}_μ, and we have

$$L_e = L_e(u_\mu, u_{\mu|\nu}) \tag{37}$$

$$L_g = \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}) \tag{38}$$

and

$$L_{\text{int}} = \operatorname{tr}(F_{\mu\nu} e^{\mu\nu}) = kR \tag{39}$$

$$R = R^\mu_a e^a_\mu \tag{40}$$

and

$$R^\mu_a = F^{\mu'\nu}_{a'b} e^b_\nu e_{a\mu'} e^{a'\mu}. \tag{41}$$

Obviously, L_e, L_g and L_{int} are invariant under the group-couple $\mathscr{L} \times \mathscr{A}$, where the group $\mathscr{A}$ is the group of a non-homonomic transformation of the coordinates.

Equation (36) shows that the gauge field (B^{ab}_μ) and the frame field (e^a_α) commonly describe the field of metric defects.

The action functional is

$$S = \int \left[L_e - \frac{\eta}{4} \operatorname{tr}(F_{\mu\nu}F^{\mu\nu}) - k \operatorname{tr}(F_{\mu\nu}e^{\mu\nu}) \right] \varepsilon \, d^4x, \tag{42}$$

where

$$\varepsilon = \det(e_\mu^a) = [-\det(g_{\mu\nu})]^{1/2}$$

and η is a new coupled constant.

Suppose that Hamilton's principle holds, and taking the variation of the action functional with respect to e_μ^a and B_μ^{ab}, we can obtain the new equations of the elastic field and the metric defect field as follows:

$$\frac{\partial L_e}{\partial u_\mu} - \frac{1}{\varepsilon} \partial_\nu \left[\varepsilon \frac{\partial L_e}{\partial u_{\mu|\nu}} \right] = 0, \tag{43}$$

$$R_a^\mu - \tfrac{1}{2} R e_a^\mu = -\frac{1}{2K} (\eta t_a^\mu + T_a^\mu) \tag{44}$$

and

$$\eta F_{ab\|\nu}^{\mu\nu} = 2k t_{ab}^\mu + T_{ab}^\mu \tag{45}$$

where

$$t_a^\mu = -[\operatorname{tr}(F_{\sigma\lambda}F^{\sigma\mu})e_a^\lambda - \tfrac{1}{4} \operatorname{tr}(F_{\lambda\sigma}F^{\lambda\sigma})e_a^\mu] \tag{46}$$

is an energy momentum tensor of metric defects,

$$t_{ab}^\mu = -\tfrac{1}{2}[S_{ab}^\mu - S_{ac}^\lambda e_\lambda^c e_b^\mu - S_{cb}^\lambda e_\lambda^c e_a^\mu] = -e_{ab\|\nu}^{\mu\nu} \tag{47}$$

is the density of the spin flow in the field of metric defects, and

$$S_{\mu\nu}^a = S_{cd}^\lambda e_\lambda^a e_\mu^c e_\nu^d = -2\partial_{[\mu} e_{\nu]}^a - 2B_{c[\mu}^a e_{\nu]}^c \tag{48}$$

is the torsion tensor of the space-time. T_a^μ is an energy–momentum tensor of the elastic field, and

$$T_{ab}^\mu = \frac{1}{\varepsilon} \cdot \frac{\partial(\varepsilon L_e)}{\partial B_\mu^{ab}} \tag{49}$$

is the density of the spin flow in the elastic field.

Therefore, the action of metric defects represents not only the geometric effect of the metric, but the physical effect of the gauge potential. Thus, this theory is different from the dynamic theory of the continuous distribution.

6. DISCUSSION

(1) In our treatment, the space-time is a non-Riemannian space of four dimensions, the geometry of the space-time is connected with metric defects, and the gauge field is of non-Riemannian geometry. This space-time not only has curvature, but also torsion. The connection of the space-time has two parts, namely the affine and the gauge. In particular, the gauge potential describes both the defects and the effect of the physical field. Thus, our procedure is different from the three state theory of the gauge field of a continuum with dislocations and disclinations given by Duan and Duan (1986).

(2) Since

$$u_{\mu\|\nu} = \partial_\nu u_\mu - \left\{ \begin{matrix} \sigma \\ \mu \quad \nu \end{matrix} \right\} u_\sigma - B^\sigma_{\mu\nu} u_\sigma,$$

then the field equation (43) contains a coupled effect of the metric defect and the gauge potential. Therefore, (43) is different from the representation of the Cauchy strain tensor provided by Golebiewska-Lasota (1979).

(3) Equation (44) is a new field equation. It is the generalization of Einstein's equation (1917) and it describes the gauge theory of the metric defect containing the elastic field. Based on continuous distribution theory of metric defects analogized with gravity theory, and setting $t^\mu_a = 0$, we obtain

$$K = \frac{1}{16\pi k}$$

where k is the so-called Newton's constant of gravity. When the gauge potential $B^\mu_{ab} \equiv 0$, we obtain the results of continuous distribution of the dynamic metric defect given by Dong and Zhang (1985) and Sedov and Berditehevski (1967).

Kondo (1958) has analogously discussed the general relativity and Guo *et al.* (1973) have analogously discussed the gauge theory of gravitational field. The correspondence given in this paper is the generalization of the correspondence between the continuous distribution theory of defects and gravity field theory or electromagnetic field theory.

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